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# Casorati determinant solutions for the discrete Painlevé-II equation

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**Abstract.** We present a class of solutions to the discrete Painlevé-II equation for particular values of its parameters. It is shown that these solutions can be expressed in terms of Casorati determinants whose entries are discrete Airy functions. The analogy between the $\tau$ function for the discrete $P_{II}$ and that of the discrete Toda molecule equation is pointed out.

## 1. Introduction

The six Painlevé transcendents are of very common occurrence in the theory of integrable systems [1]. Nonlinear evolution equations, integrable through inverse scattering techniques, were shown to possess one-dimensional (similarity) reductions that are just Painlevé equations. This feature of integrable PDEs eventually evolved into an integrablity criterion [2], the Painlevé property being intimately linked to integrablity. Discrete integrable systems have recently become the focus of interest and an active domain of research. The study of the partition function in a 2D model of quantum gravity [3, 4] led to the discovery of the discrete analogue of the Painlevé-I ($P_I$) equation. It was followed closely afterwards by the derivation of the discrete $P_{II}$ in both a quantum gravity setting [5] and as a *similarity reduction* of a lattice version of the mKdV equation [6]. The remaining discrete Painlevé equations ($dP_{III}$ to $dP_V$) were derived [7] using a more direct approach reminiscent of the Painlevé–Gambier [8] method for the continuous ones. This method, derived in [9] and dubbed singularity confinement, is the discrete equivalent of the Painlevé approach and offers an algorithmic criterion for discrete integrability. One important result of these investigations is that the form of the discrete Painlevé equations is not unique: there exist several discrete analogues for each continuous Painlevé equations.

The continuous Painlevé equations were shown to be transcendental in the sense that their general solution cannot be expressed in terms of elementary functions [10]. In fact, this solution can be obtained only through inverse scattering methods. However, in some

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particular cases (for special values of parameters) the solution to the Painlevé equations can be expressed in terms of special functions [11–13]. For example, $P_{II}$

$$w_{xx} - 2w^3 + 2xw + \alpha = 0 \tag{1}$$

has a solution for $\alpha = -(2N+1)$

$$w = \left(\log \frac{\tau_{N+1}}{\tau_N}\right)_x \tag{2}$$

where $\tau_N$ is given by an $N \times N$ Wronskian of the Airy function

$$\tau_N = \begin{vmatrix} \mathrm{Ai} & \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ai} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N-1}\mathrm{Ai} \\ \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ai} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2}\mathrm{Ai} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N}\mathrm{Ai} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N-1}\mathrm{Ai} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N}\mathrm{Ai} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2N-2}\mathrm{Ai} \end{vmatrix}. \tag{3}$$

Note that Ai is the Airy function, satisfying

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Ai} = x\,\mathrm{Ai}. \tag{4}$$

From the close analogy that is known to exist between the continuous and discrete Painlevé equations, one would also expect special function-like solutions to exist for the discrete Painlevé equations, and this is indeed the case. As was shown in [14], $dP_{II}$ has elementary solutions that can be expressed in terms of the discrete equivalent to the Airy function. In [14] only the simplest of these solutions was derived explicitly. The method for obtaining the higher ones was based on the existence of an auto-Bäcklund transform for $dP_{II}$, but it is not clear how one can obtain the general expression for these special function solutions following this method. In this paper we intend to present the answer to this problem. Using Hirota's bilinear formalism, we show that these particular solutions to $dP_{II}$ can be written as Casorati determinants whose entries are the discrete analogues of the Airy function.

## 2. Special solutions of $dP_{II}$

We consider $dP_{II}$

$$w_{n+1} + w_{n-1} = \frac{(\alpha n + \beta)w_n + \gamma}{1 - w_n^2} \tag{5}$$

where $\alpha$, $\beta$ and $\gamma$ are arbitrary constants. First, let us seek a simple solution of (5). It is easily shown that if $w_n$ satisfies the Riccati-type equation

$$w_{n+1} = \frac{w_n - (an + b)}{1 + w_n} \tag{6}$$

then it gives a solution of (5) with the constraint $\gamma = -\alpha/2$. In fact, we have from (6)

$$w_{n-1} = \frac{w_n + (an - a + b)}{1 - w_n}. \tag{7}$$

Adding (6) and (7), we obtain

$$w_{n+1} + w_{n-1} = \frac{(2an - a + 2b + 2)w_n - a}{1 - w_n^2} \tag{8}$$

which is a special case of (5). Now we put

$$w_n = \frac{F_n}{G_n} \tag{9}$$

and substitute (9) into (6) and, assuming that the numerators and the denominators of both sides of (6) are equal, respectively, we have

$$F_{n+1} = F_n - (an + b)G_n \tag{10a}$$

$$G_{n+1} = G_n + F_n. \tag{10b}$$

Eliminating $F_n$ from (10$a$) and (10$b$), we see that $G_n$ satisfies

$$G_{n+2} - 2G_{n+1} + G_n = -(an + b)G_n \tag{11}$$

which is considered to be the discrete version of (4) and has a solution given by the discrete analogue of the Airy function. By means of the solution, $w_n$ is expressed as

$$w_n = \frac{G_{n+1}}{G_n} - 1. \tag{12}$$

It is possible to construct a series of solutions expressed by the discrete analogue of the Airy function. We here give the result, leaving the derivation until the next section. We consider the $\tau$ function

$$\tau_N^n = \begin{vmatrix} A_n & A_{n+2} & \cdots & A_{n+2N-2} \\ A_{n+1} & A_{n+3} & \cdots & A_{n+2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+N-1} & A_{n+N+1} & \cdots & A_{n+3N-3} \end{vmatrix} \tag{13}$$

where $A_n$ satisfies

$$A_{n+2} = 2A_{n+1} - (pn + q)A_n. \tag{14}$$

We can show that $\tau_N^n$ satisfies the following bilinear forms

$$\tau_{N+1}^{n-1}\tau_{N-1}^{n+2} = \tau_N^{n-1}\tau_N^{n+2} - \tau_N^n\tau_N^{n+1} \tag{15}$$

$$\tau_{N+1}^{n+2}\tau_N^{n+1} - 2\,\tau_{N+1}^{n+1}\tau_N^{n+2} + (pn + q)\,\tau_{N+1}^{n}\tau_N^{n+3} = 0 \tag{16}$$

and

$$\tau_{N+1}^{n+1}\tau_{N-1}^{n+2} = -(p(n+2N)+q)\ \tau_N^{n+2}\tau_N^{n+1} + (pn+q)\ \tau_N^n\tau_N^{n+3}. \tag{17}$$

Applying the dependent variable transformation as

$$w_n = \frac{\tau_{N+1}^{n+1}\tau_N^n}{\tau_{N+1}^n\tau_N^{n+1}} - 1 \tag{18}$$

we obtain a special case of $dP_{II}$

$$w_{n+1} + w_{n-1} = \frac{(2pn+(2N-1)p+2q)\ w_n - (2N+1)p}{1-w_n^2}. \tag{19}$$

We note that (8) and its solution is recovered by putting $p = a$, $q = b+1$, and $N = 0$. We also note that (19) reduces to (1) with $\alpha = -(2N+1)$ if we choose $p = -\epsilon^3$, $q = 1$, $w_n = \epsilon w$ and $n = x/\epsilon$, and take the limit $\epsilon \to 1$.

## 3. Derivation of the results

In this section we show that (13) really gives the solution of (19) through the dependent variable transformation (18).

First, let us prove that the $\tau$ function (13) satisfies the bilinear forms (15)–(17). For this purpose we show that (15)–(17) reduce to the Jacobi identity or the Plücker relations. Before doing so, we give a brief explanation of the Jacobi identity. Let $D$ be some determinant, and $D\binom{i}{j}$ be the determinant with the $i$th row and the $j$th column removed from $D$. Then the Jacobi identity is given by

$$D\begin{pmatrix} i \\ j \end{pmatrix}\ D\begin{pmatrix} k \\ l \end{pmatrix} - D\begin{pmatrix} i \\ l \end{pmatrix}\ D\begin{pmatrix} k \\ j \end{pmatrix} = DD\begin{pmatrix} i & k \\ j & l \end{pmatrix}. \tag{20}$$

It is easily seen that (15) is nothing but the Jacobi identity. In fact, taking $\tau_{N+1}^{n-1}$ as $D$, and putting $i = j = 1$, $k = l = N+1$, we find that (15) reduces to (20). Hence it is shown that (13) satisfies (15).

Let us next prove that (16) is true. Notice that $\tau_N^n$ is rewritten as

$$\begin{aligned}
\tau_N^n &= \begin{vmatrix} A_n & \cdots & A_{n+2N-4} & 2A_{n+2N-3} - (p(n+2N-4)+q)A_{n+2N-4} \\ A_{n+1} & \cdots & A_{n+2N-3} & 2A_{n+2N-2} - (p(n+2N-3)+q)A_{n+2N-3} \\ \vdots & \ddots & \vdots & \vdots \\ A_{n+N-1} & \cdots & A_{n+3N-5} & 2A_{n+3N-4} - (p(n+3N-5)+q)A_{n+3N-5} \end{vmatrix} \\
&= \begin{vmatrix} A_n & \cdots & A_{n+2N-4} & 2A_{n+2N-3} \\ A_{n+1} & \cdots & A_{n+2N-3} & 2A_{n+2N-2} - pA_{n+2N-3} \\ \vdots & \ddots & \vdots & \vdots \\ A_{n+N-1} & \cdots & A_{n+3N-5} & 2A_{n+3N-4} - (N-1)pA_{n+3N-5} \end{vmatrix} \\
&= \begin{vmatrix} A_n & 2A_{n+1} & \cdots & 2A_{n+2N-3} \\ A_{n+1} & 2A_{n+2} - pA_{n+1} & \cdots & 2A_{n+2N-2} - pA_{n+2N-3} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+N-1} & 2A_{n+N} - (N-1)pA_{n+N-1} & \cdots & 2A_{n+3N-4} - (N-1)pA_{n+3N-5} \end{vmatrix} \\
&= 2^{N-1}\begin{vmatrix} B_n^{(0)} & A_{n+1} & \cdots & A_{n+2N-3} \\ \vdots & \vdots & \ddots & \vdots \\ B_n^{(N-1)} & A_{n+N} & \cdots & A_{n+3N-4} \end{vmatrix}.
\end{aligned} \tag{21}$$

where $B_n^{(k)}$, $k = 0, 1, \ldots$, are given by

$$B_n^{(0)} = A_n \qquad B_n^{(k)} = A_{n+k} + \frac{kp}{2} B_n^{(k-1)} \qquad \text{for } k \geqslant 1. \tag{22}$$

Similarly, we have

$$(pn+q)\tau_N^n = 2^{N-1} \begin{vmatrix} A_{n+1} & B_{n+2}^{(0)} & A_{n+3} & \cdots & A_{n+2N-3} \\ A_{n+2} & B_{n+2}^{(1)} & A_{n+4} & \cdots & A_{n+2N-2} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ A_{n+N} & B_{n+2}^{(N-1)} & A_{n+N+2} & \cdots & A_{n+3N-4} \end{vmatrix}. \tag{23}$$

Let us introduce the notation

$$j = \begin{pmatrix} A_{n+j} \\ A_{n+j+1} \\ \vdots \end{pmatrix} \qquad j' = \begin{pmatrix} B_{n+j}^{(0)} \\ B_{n+j}^{(1)} \\ \vdots \end{pmatrix} \qquad \phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{24}$$

For example, $\tau_N^n$ and $(pn+q)\tau_N^n$ are rewritten as

$$\begin{aligned} \tau_N^n &= |0, 2, \ldots, 2N-2| = |0, 2, \ldots, 2N-2, \phi| \\ &= 2^{N-1}|0', 1, 3, \ldots, 2N-3| \end{aligned}$$

$$(pn+q)\tau_N^n = 2^{N-1}|1, 2', 3, 5, \ldots, 2N-3| = 2^{N-1}|1, 2', 3, \ldots, 2N-3, \phi|$$

respectively. Now consider the following identity of the $(2N+2)\times(2N+2)$ determinant

$$0 = \left|\begin{array}{cc|ccc|ccc|cc} -1 & 0' & 1 & \cdots & 2N-5 & & \emptyset & & 2N-3 & \phi \\ \hline -1 & 0' & & \emptyset & & 1 & \cdots & 2N-5 & 2N-3 & \phi \end{array}\right|. \tag{25}$$

Applying the Laplace expansion on the right-hand side of (25), we obtain

$$\begin{aligned} 0 = |-1, 0', 1, \ldots, 2N-5| &\times |1, \ldots, 2N-5, 2N-3, \phi| \\ &- |-1, 1, \ldots, 2N-5, 2N-3| \times |0', 1, \ldots 2N-5, \phi| \\ &+ |-1, 1, \ldots, 2N-5, \phi| \times |0', 1, \ldots, 2N-5, 2N-3| \end{aligned} \tag{26}$$

which is nothing but the special case of the Plücker relations. Equation (26) is rewritten by using (21) and (23) as

$$0 = (p(n-2)+q)\ \tau_N^{n-2}\ \tau_{N-1}^{n+1} - 2\ \tau_N^{n-1}\ \tau_{N-1}^n + \tau_N^n\ \tau_{N-1}^{n-1} \tag{27}$$

which is essentially the same as (16).

We next prove that (17) holds. We have the following equation similar to (21) and (23)

$$(p(n+2N)+q)\tau_N^{n+2} = -|2, \ldots, 2N-2, 2N+2| + 2^{N-1}|2', 3, \ldots, 2N-3, 2N+1|. \tag{28}$$

Then the right-hand side of (17) is rewritten as

$$\begin{aligned}|2,\ldots,2N-2,2N+2|\times|1,3,\ldots,2N-1|\\ -2^{N-1}|2',3,\ldots,2N-3,2N+1|\times|1,3,\ldots,2N-1|\\ +2^{N-1}|1,2',3,\ldots,2N-5,2N-3|\times|3,5,\ldots 2N-1,2N+1|.\end{aligned} \tag{29}$$

From the identity

$$\begin{aligned}0&=\left|\begin{array}{cc|ccc|ccc|cc}1&2'&3&\cdots&2N-3&&\emptyset&&2N-1&2N+1\\ \hline 1&2'&&\emptyset&&3&\cdots&2N-3&2N-1&2N+1\end{array}\right|\\ &=|1,2',3,\ldots,2N-3|\times|3,5,\ldots 2N-5,2N-1,2N+1|\\ &\quad-|1,3,\ldots 2N-3,2N-1|\times|2',3,\ldots,2N-3,2N+1|\\ &\quad+|1,3,\ldots,2N-3,2N+1|\times|2',3,\ldots 2N-3,2N-1|\end{aligned} \tag{30}$$

the second and third terms of (29) yield

$$\begin{aligned}-2^{N-1}|1,3,\ldots,2N-3,2N+1|\times|2',3,\ldots,2N-3,2N-1|\\ =-|1,3,\ldots,2N-3,2N+1|\times|2,4,\ldots,2N-2,2N|.\end{aligned} \tag{31}$$

Hence, equation (17) is reduced to

$$\begin{aligned}|2,4,\ldots,2N-2,2N+2|\times|1,3,\ldots,2N-3,2N-1|\\ -|1,3,\ldots,2N-3,2N+1|\times|2,4,\ldots,2N-2,2N|\\ =|1,3,\ldots,2N-1,2N+1|\times|2,4,\ldots,2N-2|\end{aligned} \tag{32}$$

which is again nothing but the Jacobi identity (20). In fact, taking $D = |1,3,\ldots,2N-1,2N+1|$, $i=1$, $j=N+1$, $k=N$ and $l=N+1$, we see that (20) is the same as (32). This completes the proof that the $\tau$ function (13) satisfies the bilinear forms (15)–(17).

Finally, let us derive (19) from the bilinear forms (15)–(17). We introduce the dependent variables by

$$v_N^n=\frac{\tau_{N+1}^n}{\tau_N^n}\qquad u_N^n=\frac{\tau_N^n\tau_N^{n+3}}{\tau_N^{n+1}\tau_N^{n+2}}. \tag{33}$$

Then (15)–(17) are rewritten as

$$v_N^{n-1}=v_{N-1}^{n+2}\left(1-\frac{1}{u_N^{n-1}}\right) \tag{34}$$

$$v_N^{n+2}-2v_N^{n+1}+(pn+q)u_N^n v_N^n=0 \tag{35}$$

$$v_N^{n+1}=v_{N-1}^{n+2}\left(-(p(n+2N)+q)+(pn+q)u_N^n\right) \tag{36}$$

respectively. Eliminating $u_N$ and $v_{N-1}$ from (34)–(36) and introducing $w_n$ defined by

$$w_n=\frac{v_N^{n+1}}{v_N^n}-1 \tag{37}$$

we obtain (19).

## 4. Concluding remarks

In this paper, we have discussed the solution of $dP_{II}$, (for semi-integer values of the parameter $\gamma/\alpha$) and shown that it can be expressed as a Casorati determinant of the discrete Airy function. The most remarkable result is the structure of the $\tau$ function (13). The subscript of $A_n$ does not vary in the same way in the horizontal and vertical directions: it increases by *one* with each new row and by *two* with each new column. This is a feature which has not been encountered before in other discrete integrable systems.

Before concluding let us point out the relation with the Toda molecule equation. It is known in general that the $\tau$ function of $P_{II}$ satisfies the Toda molecule equation [11]

$$\frac{d^2}{dx^2}\tau_N\cdot\tau_N-\left(\frac{d}{dx}\tau_N\right)^2=\tau_{N+1}\tau_{N-1}\qquad N=0,1,2,\ldots \tag{38}$$

whose solution is expressed as

$$\tau_N=\begin{vmatrix} f & \frac{d}{dx}f & \cdots & \left(\frac{d}{dx}\right)^{N-1}f \\ \frac{d}{dx}f & \left(\frac{d}{dx}\right)^2 f & \cdots & \left(\frac{d}{dx}\right)^N f \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{d}{dx}\right)^{N-1}f & \left(\frac{d}{dx}\right)^N f & \cdots & \left(\frac{d}{dx}\right)^{2N-2}f \end{vmatrix} \tag{39}$$

where $f$ is an arbitrary function. It is clear that (3) is a special case of (39). Hence, we may expect that the $\tau$ function of $dP_{II}$ satisfies the discrete Toda molecule equation proposed by Hirota [15]

$$\Delta^2\tau_N^n\cdot\tau_N^n-\left(\Delta\tau_N\right)^2=\tau_{N+1}^n\tau_{N-1}^{n+2}\qquad N=0,1,2,\ldots \tag{40a}$$

or

$$\tau_N^{n+2}\tau_N^n-\left(\tau_N^{n+1}\right)^2=\tau_{N+1}^n\tau_{N-1}^{n+2}\qquad N=0,1,2,\ldots \tag{40b}$$

whose solution is given by

$$\tau_N=\begin{vmatrix} f_n & \Delta f_n & \cdots & \Delta^{N-1}f_n \\ \Delta f_n & \Delta^2 f_n & \cdots & \Delta^N f_n \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{N-1}f_n & \Delta^N f_n & \cdots & \Delta^{2N-2}f_n \end{vmatrix} \tag{41}$$

for arbitrary $f_n$, where $\Delta$ is a forward difference operator in $n$ defined by

$$\Delta\,\tau_N^n=\tau_N^{n+1}-\tau_N^n.$$

However, because of the difference in the structure of the $\tau$ function mentioned above, that of $dP_{II}$ does not satisfy the discrete Toda molecule equation (40) itself. In fact, equation (15) may be regarded as an alternative of (40), which is rewritten as

$$\left(\Delta\Delta'\tau_N^n\right)\cdot\tau_N^n-\left(\Delta\tau_N^n\right)\cdot\left(\Delta'\tau_N^n\right)=\tau_{N+1}^n\tau_{N-1}^{n+3} \tag{43}$$

where $\Delta'$ is given by

$$\Delta' \tau_N^n = \tau_N^{n+2} - \tau_N^n.$$

Indeed, equation (43) also reduces to the ordinary Toda molecule equation (38) in the continuum limit.

It is expected that the other discrete Painlevé equations also have solutions expressed by Casorati determinants whose entries are the discrete special functions. In particular for $dP_{III}$ it was shown in [16] that solutions in terms of discrete Bessel functions exist for some values of the parameters, while for $dP_{IV}$ the particular solutions are in terms of discrete parabolic cylinder (Weber–Hermite) functions [17]. In a forthcoming paper we intend to present Casorati determinant-type solutions for these discrete Painlevé equations. One more interesting point concerns the existence of rational solutions. Both continuous and discrete Painlevé equations possess such solutions, and in principle it should be possible to obtain general expressions for them in terms of Casorati determinants.

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## References

[1] Levi D and P Winternitz 1992 (ed) *Painlevé Transcendents (NATO ASI series B)* (New York: Plenum)
[2] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333
[3] Brézin E and Kazakov V A 1990 *Phys. Lett.* **236B** 144
[4] Its A R, Kitaev A V and Fokas A S 1990 *Usp. Mat. Nauk* **45** 135
[5] Periwal V and Shevitz D 1990 *Phys. Rev. Lett.* **64** 135
[6] Nijhoff F and Papageorgiou V G 1991 *Phys. Lett.* **153A** 337
[7] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829
[8] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[9] Grammaticos B, Ramani A and Papageorgiou V G 1991 *Phys. Rev. Lett.* **67** 1825
[10] Painlevé P 1902 *Acta. Math.* **25** 1
[11] Okamoto K 1987 *Ann. Mat. Pura Appl.* **146** 337; 1987 *Japan J. Math.* **13** 47; 1986 *Math. Ann.* **275** 221; 1987 *Funkcial. Ekvac.* **30** 305
[12] Nakamura A 1992 *J. Phys. Soc. Japan* **61** 3007
[13] Gromak V A and Lukashevich N A 1990 *The Analytic Solutions of the Painlevé Equations* (Minsk: Universiteskoye Publishers) (in Russian)
[14] Ramani A and Grammaticos B 1992 *J. Phys. A: Math. Gen.* **25** L633
[15] Hirota R 1987 *J. Phys. Soc. Japan* **56** 4285
[16] Grammaticos B, Nijhoff F, Papageorgiou V G, Ramani A and Satsuma J 1993 *Phys. Lett.* A submitted
[17] Tamizhmanni K M, Ramani A and Grammaticos B 1994 *Lett. Math. Phys.* to appear